26. $y' = e^{-y}x^2 + \frac{1}{x} - e^{y}$ $\qquad y_1 = \ln x$

$z = e^{y} \longrightarrow z' = y'e^{y}$, $y'e^{y} = x^2 + \frac{e^{y}}{x} - e^{2y}$

$z' - \frac{z}{x} + z^2 = x^2 \longrightarrow$ معادله ریکاتی

$z = z_1 + \frac{1}{t}$, $z_1 = e^{y_1} = e^{\ln x} = x \longrightarrow z = x + \frac{1}{t}$

$z' = 1 - \frac{t'}{t^2}$

$1 - \frac{t'}{t^2} - \frac{1}{x}\left(x + \frac{1}{t}\right) + \left(x + \frac{1}{t}\right)^2 = x^2 \longrightarrow \cancel{1} - \frac{t'}{t^2} - \cancel{1} - \frac{1}{xt} + \cancel{x^2} + \frac{1}{t^2} + \frac{2x}{t} = \cancel{x^2}$

$\frac{-t'}{t^2} - \frac{1}{xt} + \frac{1}{t^2} + \frac{2x}{t} = 0 \longrightarrow t' + \left(\frac{1}{x} - 2x\right)t = 1 \longrightarrow$ معادله مرتبه اول خطی

$t = e^{\int(-\frac{1}{x}+2x)dx}\left(\int e^{\int(\frac{1}{x}-2x)dx}dx + c\right) = e^{-\ln x + x^2}\left(\int e^{\ln x - x^2}dx + c\right)$

$= \frac{1}{x}e^{x^2}\left(\int x e^{-x^2}dx + c\right) = \frac{1}{x}e^{x^2}\left(-\frac{1}{2}e^{-x^2} + c\right) = \frac{-1 + ce^{x^2}}{2x}$

$\exp(y(x)) = z = x + \frac{1}{t} = x + \frac{2x}{-1 + ce^{x^2}} = \frac{x + cxe^{x^2}}{-1 + ce^{x^2}}$

$g(x) = \frac{\overset{c'}{2c}e^{x^2} + 1}{\underset{c'}{2c}e^{x^2} - 1} \longrightarrow \exp(y_c(x)) = xg(x)$ $\qquad$ گزینه 4

27. $y_1(x)=\sin x \qquad y_2(x)=?$

$$W=\begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}=\begin{vmatrix} \sin x & y_2(x) \\ \cos x & y_2'(x) \end{vmatrix}=y_2'(x)\sin x-y_2\cos x$$

$(0,\frac{\pi}{2})$ در بازه $\Longrightarrow y_2'(x)\sin x-y_2(x)\cos x=\sin^2 x$

$y_2'(x)-y_2(x)\cot x=\sin x \longrightarrow$ معادله مرتبه اول خطی

$$y_2(x)=e^{\int\cot x\,dx}\left(\int \sin x\, e^{\int \cot x\,dx}dx+c\right)$$

$$=e^{\ln|\sin x|}\left(\int \sin x\, e^{-\ln|\sin x|}dx+c\right)$$

$$=\sin x\left(\int dx+c\right)=\sin x\,(x+c)=x\sin x+c\sin x$$

که با توجه به گزینه‌ها حاصل برابر با $x\sin x$ خواهد شد. گزینه 3

28. $\int x^3 J_0(x)\,dx$

$$\int u\,dv = uv - \int v\,du \longrightarrow \begin{cases} u = x^2 \\ dv = xJ_0(x) \end{cases} \longrightarrow \begin{cases} du = 2x\,dx \\ v = xJ_1(x) \end{cases}$$

$$\begin{aligned}\int x^3 J_0(x)\,dx &= x^3 J_1(x) - 2\int x^2 J_1(x)\,dx \\ &= x^3 J_1(x) - 2x^2 J_2(x) + C \\ &= x^2\left(xJ_1(x) - 2J_2(x)\right) + C\end{aligned}$$

با توجه به گزینه ها، گزینه 1 درست است.

29. $f(t)=t^n e^{-t}$, $g(t)=e^t \dfrac{d^n f(t)}{dt^n}$

$$\mathcal{L}\left\{e^t \underbrace{\frac{d^n f(t)}{dt^n}}_{P(t)}\right\}=P(s-1)$$

$$P(s)=\mathcal{L}\left\{\frac{d^n f(t)}{dt^n}\right\}=\mathcal{L}\left\{t^n e^t\right\}=\frac{s^n n!}{(s+1)^{n+1}}$$

$$P(s-1)=\frac{(s-1)^n n!}{s^{n+1}}$$

گزینه $\underline{2}$

30. $\int_0^1 \frac{y(xt)}{\sqrt{1-t}}\,dt=\sqrt{x}$ $\qquad xt=u \to x\,dt=du$

$$\int_0^x \frac{y(u)}{\sqrt{1-\frac{u}{x}}}\frac{du}{x}=\sqrt{x}=\int_0^x \frac{y(u)}{\frac{\sqrt{x-u}}{\sqrt{x}}}\frac{du}{x}=\sqrt{x}$$

$$\Rightarrow \frac{1}{\sqrt{x}}\underbrace{\int_0^x \frac{y(u)}{\sqrt{x-u}}\,du}_{y(x)*\frac{1}{\sqrt{x}}}=\sqrt{x}$$

$$\frac{1}{\sqrt{x}}\,y(x)*\frac{1}{\sqrt{x}}=\sqrt{x}\longrightarrow y(x)*\frac{1}{\sqrt{x}}=x$$

$$\xrightarrow{\mathcal{L}} Y(s)\frac{\Gamma(\frac{1}{2})}{\sqrt{s}}=\frac{1}{s^2}\longrightarrow Y(s)=\frac{1}{\sqrt{\pi}}\,\frac{1}{s^{\frac{3}{2}}}$$

$*$ $\Gamma(\frac{1}{2})=\sqrt{\pi}$, $\Gamma(p+1)=p\,\Gamma(p)$

$$Y(s)=\frac{1}{\sqrt{\pi}}\,\frac{1}{s^{\frac{3}{2}}}\,\frac{\Gamma(\frac{3}{2})}{\Gamma(\frac{3}{2})}\xrightarrow{\mathcal{L}^{-1}}\frac{1}{\sqrt{\pi}}\,x^{\frac{1}{2}}\,\frac{1}{\frac{1}{2}\sqrt{\pi}}=\frac{2}{\pi}\sqrt{x}$$

$$\longrightarrow y(x)=\frac{2}{\pi}\sqrt{x}$$

جواب صحیح در گزینه ها نیست.