(27)

$x^3 y''' - x^2 y'' + 2xy' - 2y = 0 \qquad x > 0 \qquad y(1) = y'(1) = y''(1)$

$y''(\frac{1}{3}) = -1$

$y'(e) = ?$

$t = \ln x$

$\to D(D-1)(D-2)y - D(D-1)y - 2y = 0$

$[D(D-1)(D-2) - D(D-1) - 2]y = 0 \to [(D-1)^2(D-2)]y = 0$

$y = c_1 e^{2t} + c_2 e^{t} + c_3 t e^{t} \xrightarrow{t = \ln x}$ ~~$y = \dots$~~

$y = c_1 x^2 + c_2 x + c_3 x \ln x \to c_1 + c_2 = y(1)$ ①

$y' = 2c_1 x + c_2 + c_3(\ln x + 1) \qquad 2c_1 + c_2 + c_3 = y'(1)$ ② , $y''(1) = 2c_1 + c_3$ ③

$y'' = 2c_1 + \frac{c_3}{x} \to 2c_1 + \frac{c_3}{\frac{1}{3}} = -1 \to \boxed{2c_1 + 3c_3 = -1}$ ④

①②③④ $\to \quad c_1 = 1 \quad c_2 = 0 \quad c_3 = -1$

$y = x^2 - x\ln x \to y(e) = e^2 - e = e(e-1)$

$y' = 2x - [\ln x + 1] \to y'(e) = 2e - 2 = 2(e-1)$

گزینه ۳ =

(۲۸) $ty'' - ty' + y = ۱ \qquad y'(۰) = ۳ \qquad y(۰) = ۱$

$\rightarrow L[ty''] - L[ty'] + L[y] = L[۱]$

$\rightarrow -\dfrac{d}{ds}\left(s^۲ Y - s\,y(۰) - y'(۰)\right) + \dfrac{d}{ds}\left(sY - y(۰)\right) + Y = \dfrac{۱}{s}$

$\rightarrow -\dfrac{d}{ds}\left(s^۲ Y - s - ۳\right) + \dfrac{d}{ds}\left(sY - ۱\right) + Y = \dfrac{۱}{s}$

$\rightarrow -[۲sY + s^۲ Y' - ۱] + [sY' + Y] + Y = \dfrac{۱}{s}$

$(s - s^۲)Y' + (۲ - ۲s)Y = \dfrac{۱ - s}{s} \rightarrow \underbrace{Y'(s) + \dfrac{۲}{s} Y(s) = \dfrac{۱}{s^۲}}$

معادله خطی مرتبه اول

$Y(s) = \dfrac{۱}{s^۲}\left\{\int \dfrac{\frac{۱}{s^۲}}{\frac{۱}{s^۲}}\,ds + c\right\} = \dfrac{۱}{s^۲}\left\{s + c\right\} = \boxed{\dfrac{۱}{s} + \dfrac{c}{s^۲}}$

$y(t) = ۱ + ct \rightarrow ۱ = ۱ + ۰ \checkmark$

$y' = c \rightarrow \boxed{c = ۳}$

$Y(s) = \dfrac{۱}{s} + \dfrac{۳}{s^۲}$

$Y(۲) = \dfrac{۱}{۲} + \dfrac{۳}{۴} = \boxed{\dfrac{۵}{۴}} \checkmark$

نمره ۲

(۲۴)

$$x^2 y''(x) + p(x)\, y'(x) + x^4 y(x) = 0 \qquad x > 0 \qquad z = f(x)$$

$$y' = \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} \Rightarrow y'' = \frac{d(y')}{dx} = \frac{d}{dx}\left(\frac{dy}{dz} \cdot \frac{dz}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dz}\right) \cdot \frac{dz}{dx} + \frac{dy}{dz} \cdot \frac{d}{dx}\left(\frac{dz}{dx}\right)$$

$$= \left(\frac{dz}{dx}\right)^2 \cdot \frac{d^2y}{dz^2} + \frac{d^2z}{dx^2} \cdot \frac{dy}{dz}$$

جایگذاری $\rightarrow$ $$x^2\left[\left(\frac{dz}{dx}\right)^2 \cdot \frac{d^2y}{dz^2} + \frac{d^2z}{dx^2} \cdot \frac{dy}{dz}\right] + p(x)\left[\frac{dy}{dz} \cdot \frac{dz}{dx}\right] + x^4 y = 0$$

x^4 بر طرفین تقسیم می‌شود $\rightarrow$ $$\frac{1}{x^2}\left(\frac{dz}{dx}\right)^2 \frac{d^2y}{dz^2} + \left[\frac{1}{x^2} \frac{d^2z}{dx^2} + \frac{p(x)}{x^4} \frac{dz}{dx}\right] \frac{dy}{dz} + y = 0$$

$$\rightarrow \frac{\left(\frac{dz}{dx}\right)^2}{x^2} = c_1 \rightarrow \frac{dz}{dx} = \pm\sqrt{c_1}\, x = c_2 x \rightarrow \boxed{z = c_2 x^2 + c_3}$$

$$\frac{1}{x^2} \frac{d^2z}{dx^2} + \frac{p(x)}{x^4} \frac{dz}{dx} = c_4 \xrightarrow{z = c_2 x^2 + c_3} \frac{2c_2}{x^2} + \frac{p(x)}{x^4}(2c_2 x) = c_4$$

$$\rightarrow 2c_2 x^2 + p(x)(2c_2 x) = c_4 x^4 \rightarrow p(x) = \frac{c_4 x^4 - 2c_2 x^2}{2c_2 x} = kx^3 - x$$

گزینه ۳

۲۹ $\quad L[J_0(x)] = \dfrac{1}{\sqrt{s^2+1}} \qquad L(J_1(x)) = ?$

می دانیم $\quad J_1(x) = -J_0'(x) \rightarrow L[J_1(x)] = -L[J_0'(x)]$

$$L[f'] = sL[f] - f(0) \longrightarrow L[J_0'(x)] = sL[J_0(x)] - J_0(0) = \frac{s}{\sqrt{s^2+1}} - 1$$

$$\rightarrow L[J_1(x)] = 1 - \frac{s}{\sqrt{s^2+1}} = \frac{\sqrt{s^2+1} - s}{\sqrt{s^2+1}} \checkmark$$

گزینه ۴

۳۰

$$x^2y'' - x(\alpha+x)y' + (\beta+x^2)y = 0 \qquad y = c_1y_1(x) + c_2y_2(x)$$

$$y_2(x) = 1\,y_1(x)\ln|x| + x^2\sum_{n=1}^{\infty} b_n x^n \rightarrow 2 = r_1 = r_2 \checkmark$$

[illegible]

$$(r-2)^2 = 0$$

$$r^2 - 4r + 4 = 0$$

$$p_0 = \lim_{x\to 0} \frac{-x^2}{x^2}(\alpha+x) = -\alpha$$

$$q_0 = \lim_{x\to 0} \frac{x^2}{x^2}(\beta+x^2) = \beta$$

$$\rightarrow r^2 + (-\alpha-1)r + \beta = 0$$

$$\left\{\begin{array}{l} r^2 - (\alpha+1)r + \beta = 0 \\ r^2 - 4r + 4 = 0 \end{array}\right\} \rightarrow \begin{array}{l} \beta = 4 \\ \alpha = 3 \end{array}$$

گزینه ۳ $= \boxed{\alpha+\beta = 7}$

۳۱

$$f(x) = \begin{cases} \sin x & 0 \le x \le \pi \\ 0 & x > 0 \end{cases}$$

$$f(-x) = -f(x) \to \text{فرد}$$

$F\{f(x)\} = -2i F_s\{f(x)\} \leftarrow$ اگر $f(x)$ تابعی فرد باشد

$$\hat{f}(\omega) = -2i F_s\{f(x)\} = -2i \int_0^{\pi} \sin x \sin \omega x \, dx = -i \int_0^{\pi} [\cos(1-\omega)x - \cos(1+\omega)x] \, dx$$

$$= -i \left[\frac{\sin(1-\omega)x}{1-\omega} - \frac{\sin(1+\omega)x}{1+\omega} \right]_0^{\pi} = -i \left[\frac{\sin(1-\omega)\pi}{1-\omega} - \frac{\sin(1+\omega)\pi}{1+\omega} \right]$$

$$= -i \left[\frac{\sin \omega\pi}{1-\omega} - \frac{-\sin \omega\pi}{1+\omega} \right] = -i \left[\frac{2\sin \omega\pi}{1-\omega^2} \right] = \frac{2i \sin \omega\pi}{\omega^2 - 1}$$

آنیتا